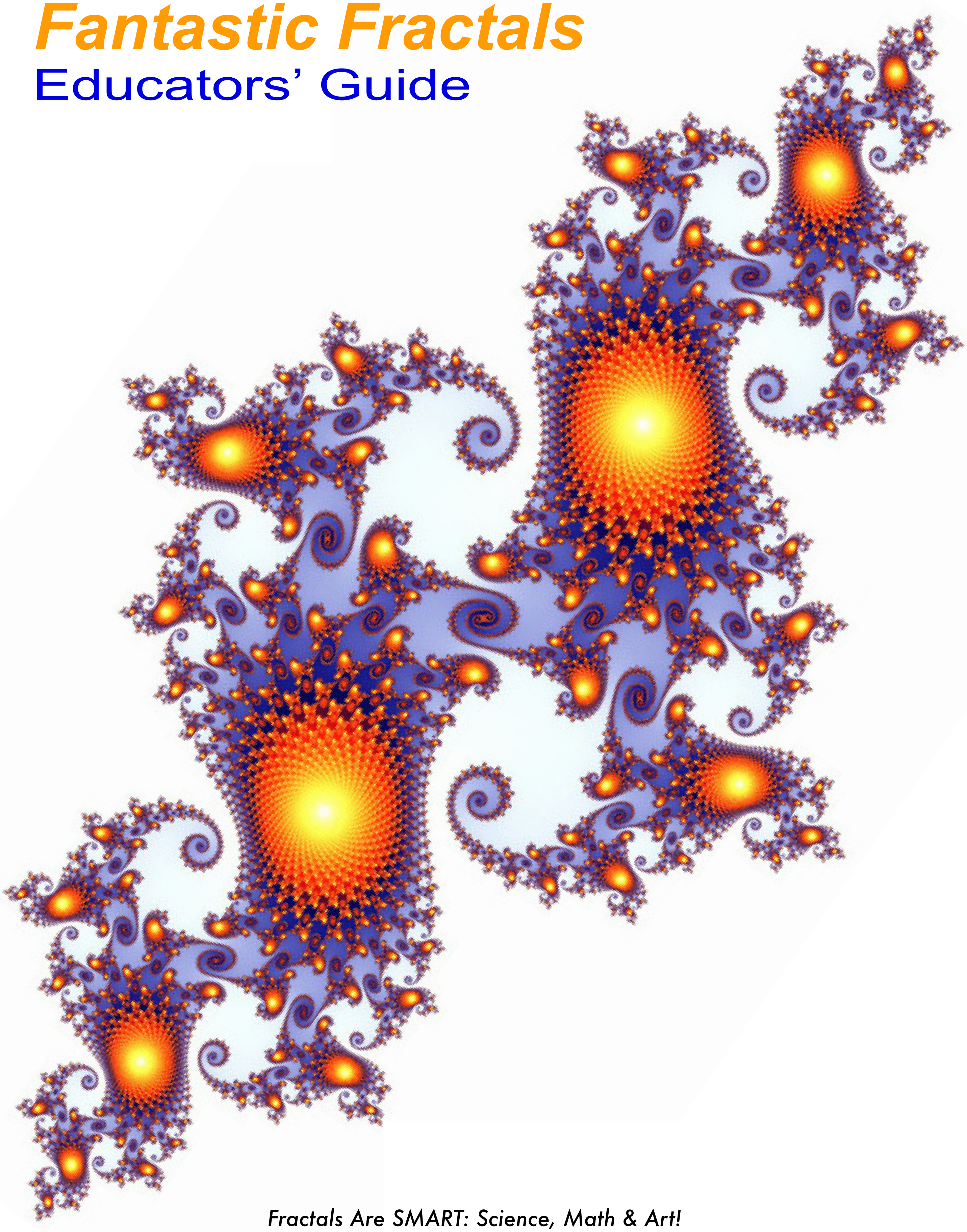
Fantastic Fractals
Educators' Guide



Fractals Are SMART: Science, Math & Art!

www.FractalFoundation.org

Contents

INTRODUCTION:

WHAT IS A FRACTAL?

A fractal is a never ending pattern that repeats itself at different scales.
This property is called “Self-Similarity.”

Fractals are extremely complex, sometimes infinitely complex - meaning you can zoom in and find the same shapes forever.

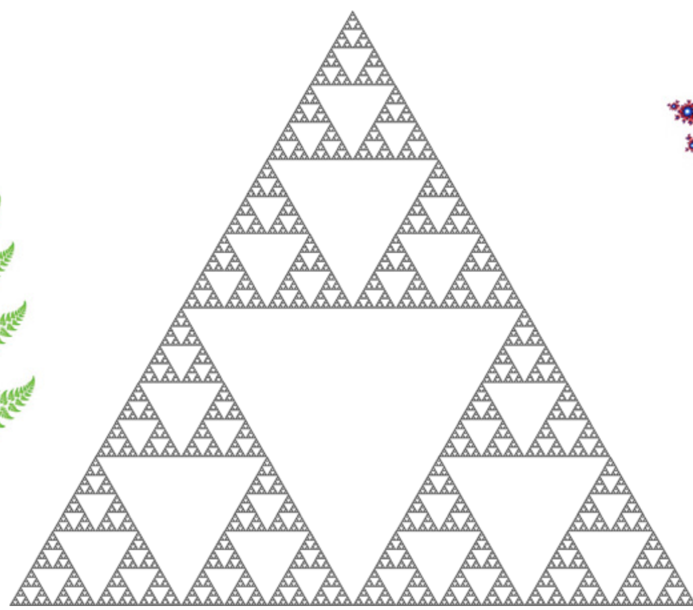
Amazingly, fractals are extremely simple to make.

A fractal is made by repeating a simple process again and again.

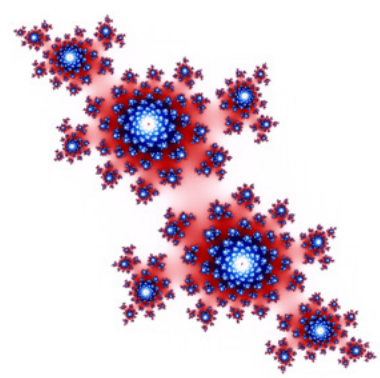
WHERE DO WE FIND FRACTALS?

In Nature

In Geometry



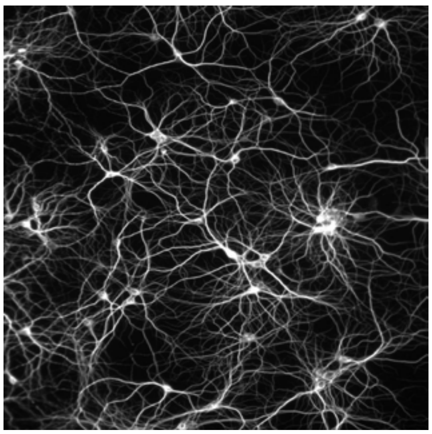
In Algebra



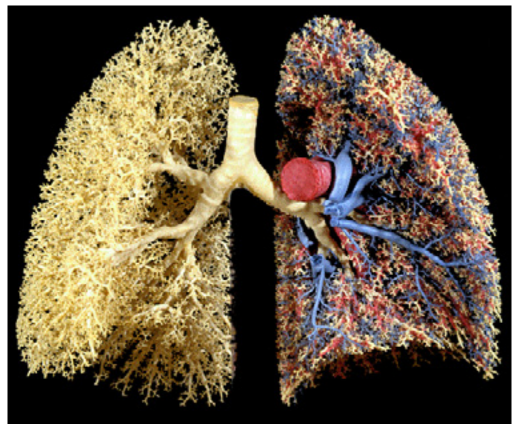
NATURAL FRACTALS

BRANCHING

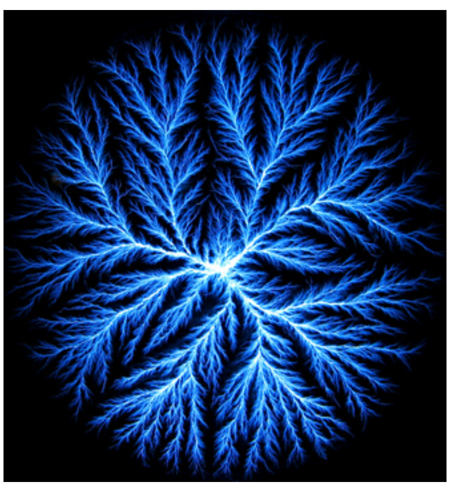
Fractals are found all over nature, spanning a huge range of scales. We find the same patterns again and again, from the tiny branching of our blood vessels and neurons to the branching of trees, lightning bolts, and river networks. Regardless of scale, these patterns are all formed by repeating a simple branching process.
A fractal is a picture that tells the story of the process that created it.



Neurons from the human cortex. The branching of our brain cells creates the incredibly complex network that is responsible for all we perceive, imagine, remember. Scale = 100 microns = 10^{-4} m.

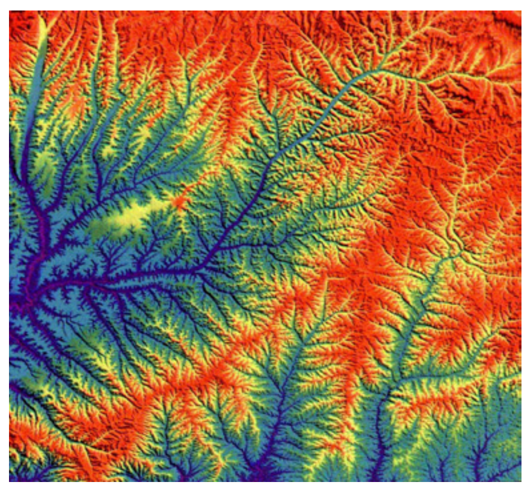


Our lungs are branching fractals with a surface area ~100 m^2. The similarity to a tree is significant, as lungs and trees both use their large surface areas to exchange oxygen and CO_2. Scale = 30 cm = $3*10^{-1}$ m.



Lichtenberg “lightning”, formed by rapidly discharging electrons in lucite. Scale = 10 cm = 10^{-1} m.

Oak tree, formed by a sprout branching, and then each of the branches branching again, etc. Scale = 30 m = $3*10^{1}$ m.



River network in China, formed by erosion from repeated rainfall flowing downhill for millions of years. Scale = 300 km = $3*10^{5}$ m.

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NATURAL FRACTALS

SPIRALS

The spiral is another extremely common fractal in nature, found over a huge range of scales. Biological spirals are found in the plant and animal kingdoms, and non-living spirals are found in the turbulent swirling of fluids and in the pattern of star formation in galaxies.

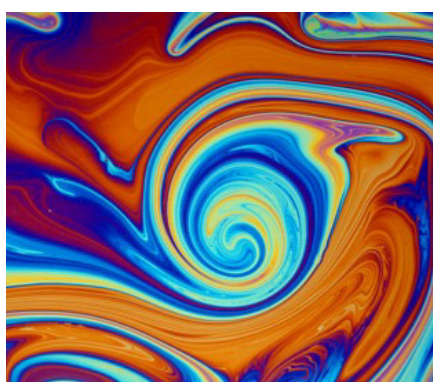
All fractals are formed by simple repetition, and combining expansion and rotation is enough to generate the ubiquitous spiral.

A fossilized ammonite from 300 million years ago. A simple, primitive organism, it built its spiral shell by adding pieces that grow and twist at a constant rate. Scale = 1 m.

A hurricane is a self-organizing spiral in the atmosphere, driven by the evaporation and condensation of sea water.
Scale = 500 km = $5*10^5$ m.

A spiral galaxy is the largest natural spiral comprising hundreds of billions of stars. Scale = 100,000 ly = $\sim10^{20}$ m.

The plant kingdom is full of spirals. An agave cactus forms its spiral by growing new pieces rotated by a fixed angle. Many other plants form spirals in this way, including sunflowers, pinecones, etc.
Scale = 50 cm = $5*10^{-1}$ m.



The turbulent motion of fluids creates spirals in systems ranging from a soap film to the oceans, atmosphere and the surface of jupiter. Scale = 5 mm = $5*10^{-3}$ m.

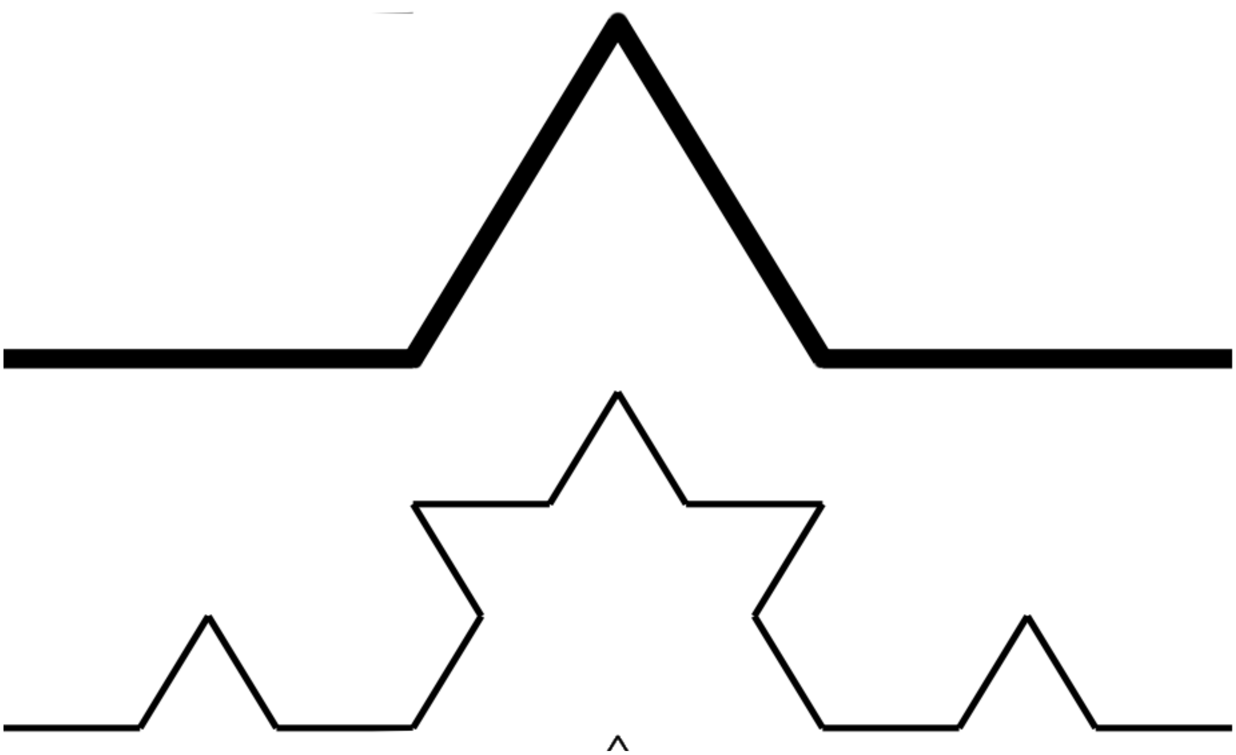
A fiddlehead fern is a self-similar plant that forms as a spiral of spirals of spirals.
Scale = 5 cm = $5*10^{-2}$ m.

GEOMETRIC FRACTALS

Purely geometric fractals can be made by repeating a simple process.

The Sierpinski Triangle is made by repeatedly removing the middle triangle from the prior generation. The number of colored triangles increases by a factor of 3 each step, 1,3,9,27,81,243,729, etc.

See the Fractivity on page 12 to learn to teach elementary school students how to draw and assemble Sierpinski Triangles.



The Koch Curve is made by repeatedly replacing each segment of a generator shape with a smaller copy of the generator. At each step, or iteration, the total length of the curve gets longer, eventually approaching infinity. Much like a coastline, the length of the curve increases the more closely you measure it.

ALGEBRAIC FRACTALS

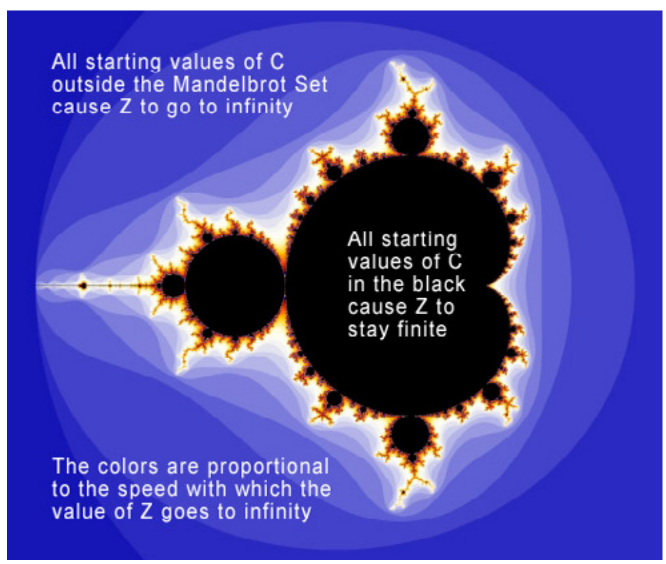
We can also create fractals by repeatedly calculating a simple equation over and over. Because the equations must be calculated thousands or millions of times, we need computers to explore them. Not coincidentally, the Mandelbrot Set was discovered in 1980, shortly after the invention of the personal computer.

HOW DOES THE MANDELBROT SET WORK?

We start by plugging a value for the variable ‘C’ into the simple equation below. Each complex number is actually a point in a 2-dimensional plane. The equation gives an answer, ‘Z_{new}’ . We plug this back into the equation, as ‘Z_{old}’ and calculate it again. We are interested in what happens for different starting values of ‘C’.

Generally, when you square a number, it gets bigger, and then if you square the answer, it gets bigger still. Eventually, it goes to infinity. This is the fate of most starting values of ‘C’. However, some values of ‘C’ do not get bigger, but instead get smaller, or alternate between a set of fixed values. These are the points inside the Mandelbrot Set, which we color black. Outside the Set, all the values of ‘C’ cause the equation to go to infinity, and the colors are proportional to the speed at which they expand.

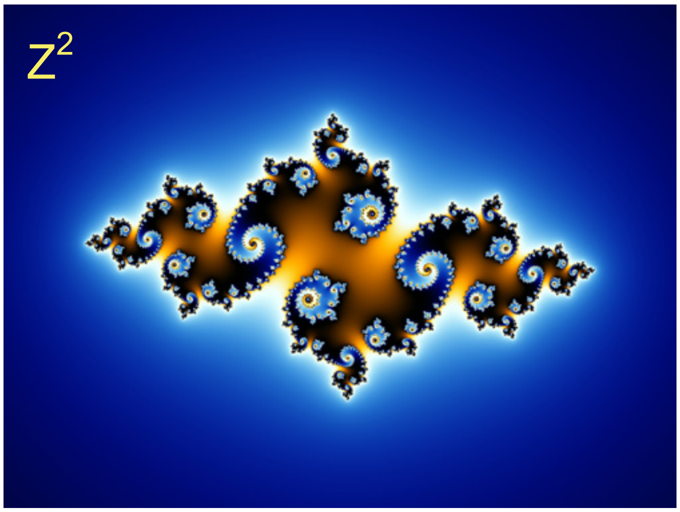
$$Z_{new} = Z_{old}^2 + C$$

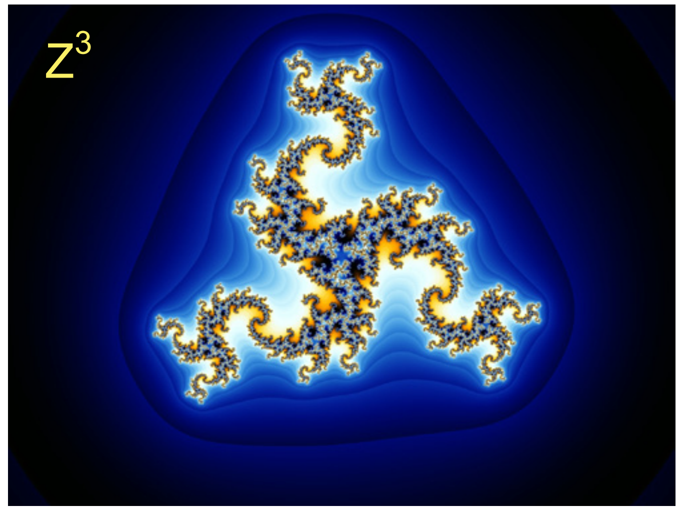


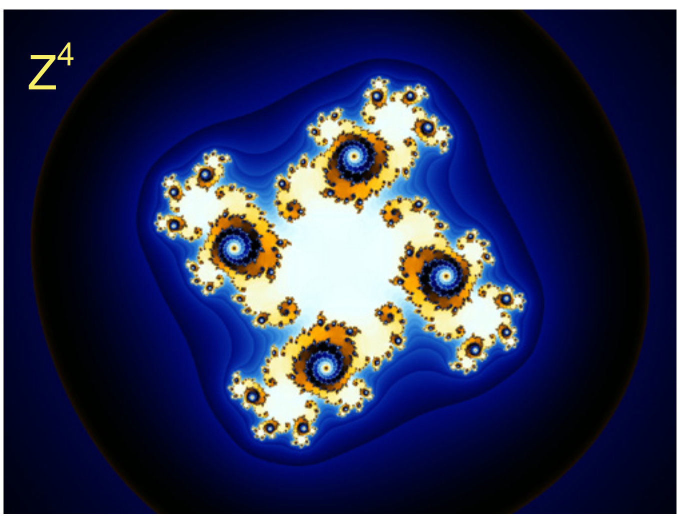
The interesting places in ths fractal are all on the edge. We can zoom in forever, and never find a clear edge. The deeper we explore, the longer the numbers become, and the slower the calculations are. Deep fractal exploration takes patience!

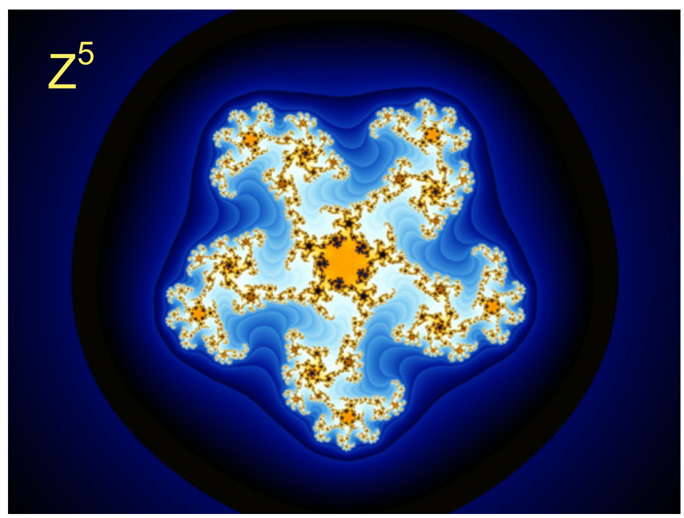
PATTERNS & SYMMETRY

The great value of fractals for education is that they make abstract math visual. When people see the intricate and beautiful patterns produced by equations, they lose their fear and instead become curious.





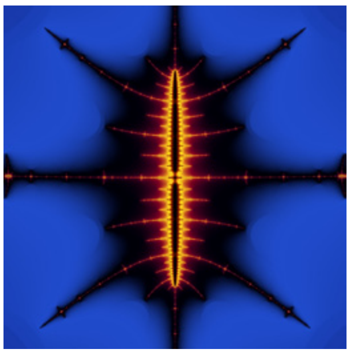
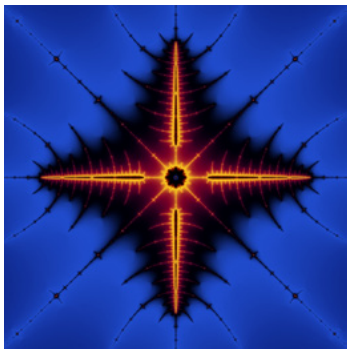
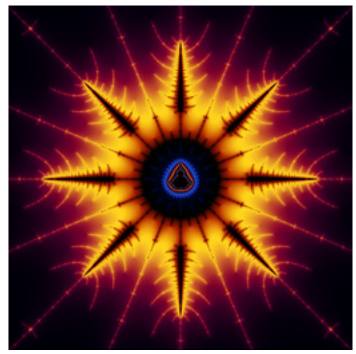




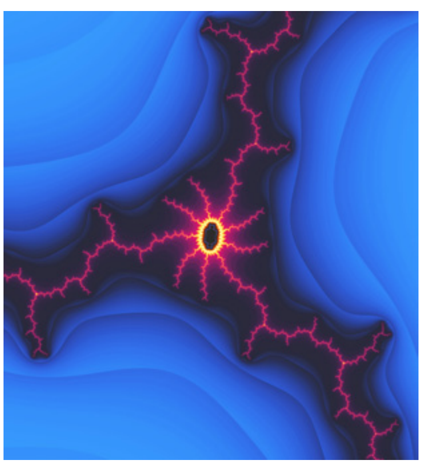
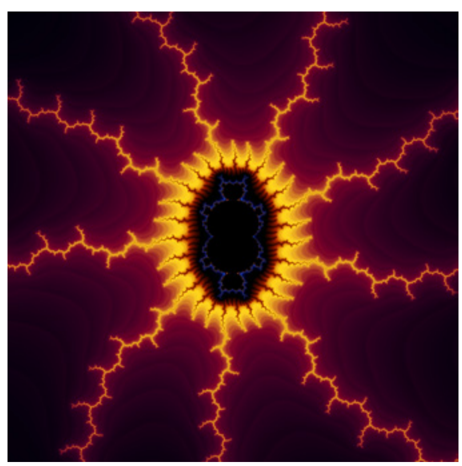
Exploring fractals is fun, and we can play with the equations to see what happens. The 4 images above are algebraic fractals known as Julia Sets. The first image in the upper left comes from the same equation as the Mandelbrot Set, $Z = Z^2 + C$. When we raise the exponent to Z^3 (i.e. Z*Z*Z), the Julia Set takes on a 3-fold symmetry, and so on. *The degree of symmetry always corresponds to the degree of the exponent.*

PATTERNS & SYMMETRY

Just as we find branching fractals in nature, we also find branching within algebraic fractals like the Mandelbrot Set. Known as “Bifurcation”, branching in these fractals is a never-ending process. The four images below are successive zooms into a detail of the $Z = Z^2 + C$ Mandelbrot Set. Two-fold symmetry branches and becomes 4-fold, which doubles into 8-fold, and then 16-fold. The branching process continues forever, and the number of arms at any level is always a power of 2.


If we explore other algebraic fractals, we find similar patterns and progressions. The two images below are details from the $Z = Z^3 + C$ Mandelbrot Set. Since the equation involves Z *cubed*, the arms now branch in 3-fold symmetry. Each of the 3 arms branches into 3 more arms, becoming 9-fold symmetry. This then trifurcates into 27 arms, 81, 243, etc. where the number of arms is always a power of 3.

Again, the educational value of fractals is that they make the behavior of equations visible. Zooming into fractals, math ceases to be intimidating, and instead becomes entrancing.

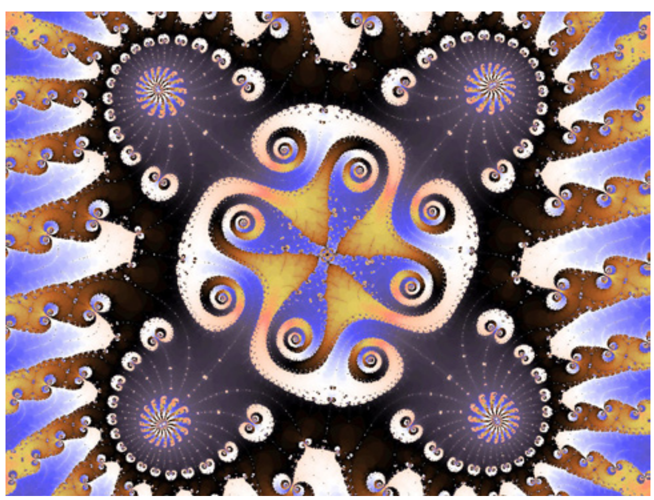
IDEAS OF SCALE

HOW BIG (OR SMALL) ARE FRACTALS?

Mathematical fractals are infinitely complex. This means we can zoom into them forever, and more detail keeps emerging. To describe the scale of fractals, we must use scientific notation:

Thousand	1,000	10^3
Million	1,000,000	10^6
Billion	1,000,000,000	10^9
Trillion	1,000,000,000,000	10^{12}
Quadrillion	1,000,000,000,000,000	10^{15}

Because of the limits of computer processors, all the fulldome fractal zooms stop at a magnification of 10^{16}. Of course the fractals keep going, but it becomes much slower to compute deeper than that. 10^{16} (or ten quadrillion) is incredibly deep. To put it in perspective, the diameter of an atom is about 10^{-10} meters, so as we zoom six orders of magnitude smaller, we're looking at things a million times smaller than an atom!
Or, to look at it another way, as we zoom into the fractals, the original object keeps growing. How big does it get when we have zoomed in 10^{16} times? The orbit of the dwarf planet Pluto is about 10^{12} meters in diameter. If we start zooming in a 10 meter dome, then the original image grows to a size larger than our entire solar system - 100,000 times larger!

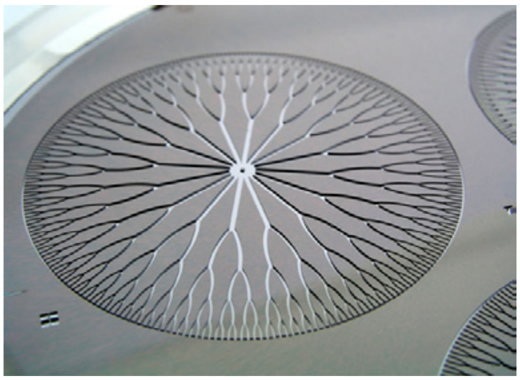


All of these zooms are just scratching the surface of the infinitely complex. Some fractals, like the Mandelbrot Set, become even more intricate and beautiful the deeper we explore. The image above exists at a depth of 10^{176} magnification!

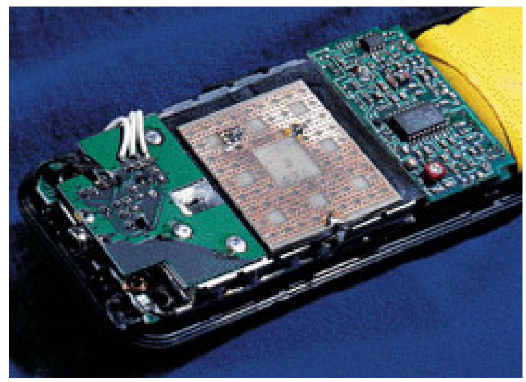
FRACTAL APPLICATIONS

A commonly asked question is: What are fractals useful for?

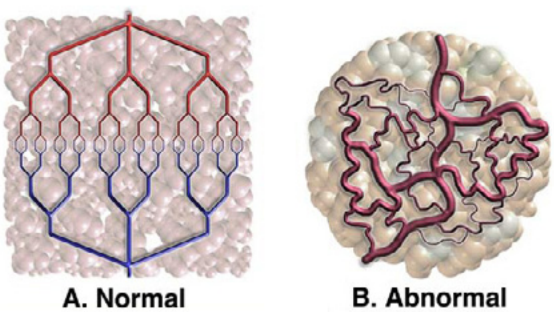
Nature has used fractal designs for at least hundreds of millions of years. Only recently have human engineers begun copying natural fractals for inspiration to build successful devices. Below are just a few examples of fractals being used in engineering and medicine.



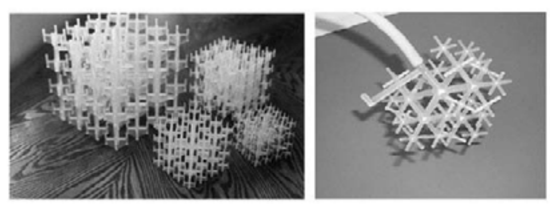
A computer chip cooling circuit etched in a fractal branching pattern. Developed by researchers at Oregon State University, the device channels liquid nitrogen across the surface to keep the chip cool.



Fractal antennas developed by Fractenna in the US and Fractus in Europe are making their way into cellphones and other devices. Because of their fractal shapes, these antennas can be very compact while receiving radio signals across a range of frequencies.

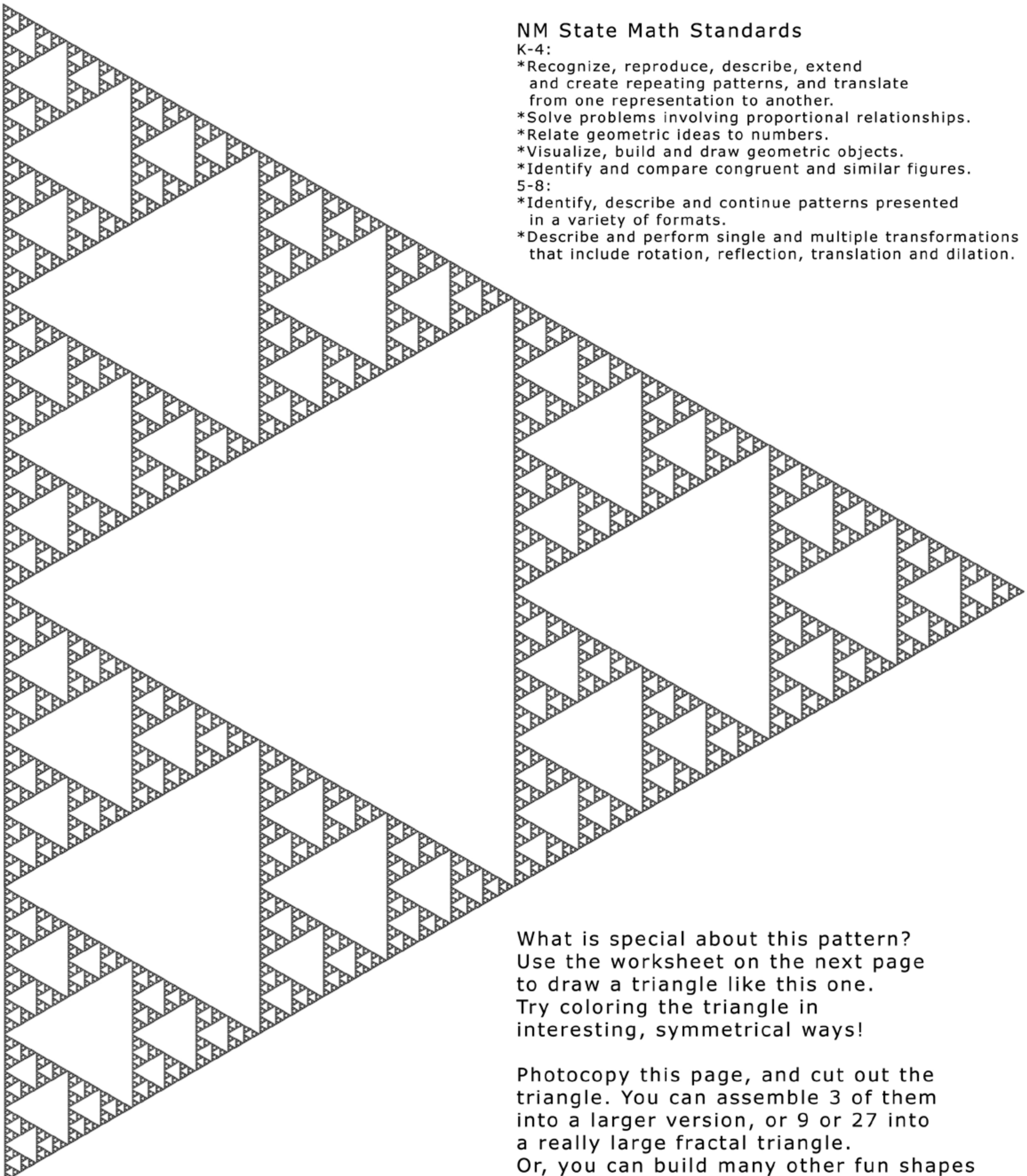


Researchers at Harvard Medical School and elsewhere are using fractal analysis to assess the health of blood vessels in cancerous tumors. Fractal analysis of CT scans can also quantify the health of lungs suffering from emphysema or other pulmonary illnesses.



Amalgamated Research Inc (ARI) creates space-filling fractal devices for high precision fluid mixing. Used in many industries, these devices allow fluids such as epoxy resins to be carefully and precisely blended without the need for turbulent stirring.

Sierpinski Triangle Fractal



NM State Math Standards
K-4:
*Recognize, reproduce, describe, extend and create repeating patterns, and translate from one representation to another.
*Solve problems involving proportional relationships.
*Relate geometric ideas to numbers.
*Visualize, build and draw geometric objects.
*Identify and compare congruent and similar figures.
5-8:
*Identify, describe and continue patterns presented in a variety of formats.
*Describe and perform single and multiple transformations that include rotation, reflection, translation and dilation.

What is special about this pattern?
Use the worksheet on the next page to draw a triangle like this one.
Try coloring the triangle in interesting, symmetrical ways!

Photocopy this page, and cut out the triangle. You can assemble 3 of them into a larger version, or 9 or 27 into a really large fractal triangle.
Or, you can build many other fun shapes with the triangles

Fractal Triangle Template

Instructions:

The three dots on the sides of the triangles are called the "midpoints" of the sides, and they are half way from one corner to the other.

Connect the dots to make a new triangle that points downward.

Color in this downward triangle.

There are then 3 triangles left over. Find the midpoints of the 3 sides of these 3 triangles, and place dots there. Connect these midpoints to make 3 smaller, downward facing triangles.

Continue the same process, making smaller and smaller triangles. For each cycle - or iteration - count the number of upward triangles, and fill in the number in the table.

Do at least 3 iterations, but fill in all the values in the table, up to the 5th.

Iteration	Number of Triangles
0	1
1	3
2	
3	
4	
5	

Fractals are never-ending patterns, and you could keep doing this for a long time. Forever, perhaps.

When you decide you're done, cut out the big triangle, flip it over, and write your name and school on the back. Join it with others to make groups of 3, 9, 27 or more triangles that make the same shape, but larger.

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Fractivity: Explore Fractals with XaoS

First, download and install the XaoS program (either Mac or Windows version) from:
http://fractalfoundation.org/resources/fractal-software/

When you run the program, it opens with an image of the Mandelbrot set.
To navigate: just point the mouse and click! On a PC, the left button zooms in and the right zooms out. On a Mac, use ctrl-click to zoom out. To pan the image around, use both buttons together, or shift-click on the Mac.

Set the defaults: From the 'Filters' menu, enable Palette Emulator. From the "Calculation" menu select Iterations, and raise it to 2000. From the 'File' menu, select Save Configuration so you don't have to make these changes again.
Color palettes are randomly generated, and can be changed with the "P" key. To cycle the colors, use "Y". There are many filters and effects to explore from the menus.

XaoS can create many different fractal types, which can be accessed by using the number keys:
Keys 1 to 5 are Mandelbrot sets with various powers. The "normal" X^2 Mandelbrot set is on key 1. (Hitting "1" is a good way to reset yourself if you get lost!)
Key 6 is a Newton fractal, exponent 3, illustrating Newton's method for finding roots to 3'd order polynomial equations.
Key 7 is the Newton fractal for exponent 4.
Key 8,9, and 0 are Barnsley fractals.
Key A - N are several other fascinating fractal formulas

Julia Sets: Every point in the Mandelbrot set (and several of the other fractals) corresponds to a unique Julia set. To explore the relationship between the Mandelbrot and Julia fractals, press "J" to enter fast-Julia mode. When you find a Julia set you like, switch over to it by pressing "M".

To save a fractal, use "File->Save Image" to save the picture for use in other prgrams. Use "File->Save" to save the actual parameters of the file, which will allow you to return to the fractal in XaoS and keep exploring it further.

Finally - use the Help file and explore the excellent tutorials! Though written by Jan Hubicka - the initial programmer - originally in Czech, they are very useful both to learn how to use the program as well as to learn about the fractals. Enjoy!

SCIENCE AND MATH EDUCATIONAL STANDARDS

From the National Council of Teachers of Mathematics (NCTM):

Recognize geometric shapes and structures in the environment and specify their location. (NCTM Geometry grades K-2)

Recognize and apply slides, flips, and turns; recognize and create shapes that have symmetry. (NCTM Geometry grades K-2)

Investigate, describe, and reason about the results of subdividing, combining, and transforming shapes. (NCTM Geometry grades 3-5)

Identify and describe line and rotational symmetry in two- and three-dimensional shapes and designs. (NCTM Geometry grades 3-5)

Recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as art, science, architecture, and everyday life.
(NCTM Geometry grades 6-8; 9-12)

Identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations. (Grades 6-8 NCTM Expectations for Algebra Knowledge)

Use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts. (Grades 9-12 NCTM Expectations for Algebra Knowledge)

From the National Science Education Standard; National Academy of Sciences:

UNIFYING CONCEPTS AND PROCESSES STANDARD:
As a result of activities in grades K-12, all students should develop understanding and abilities aligned with the following concepts and processes:

- Systems, order, and organization
- Evidence, models, and explanation
- Constancy, change, and measurement
- Evolution and equilibrium
- Form and function

Structure and function in living systems; Diversity and adaptations of organisms; Interdependence of organisms.